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Uniqueness of canonical tensor model with local time

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Abstract

Canonical formalism of the rank-three tensor model has recently been proposed, in which "local" time is consistently incorporated by a set of first class constraints. By brute-force analysis, this paper shows that there exist only two forms of a Hamiltonian constraint which satisfies the following assumptions: (i) A Hamiltonian constraint has one index. (ii) The kinematical symmetry is given by an orthogonal group. (iii) A consistent first class constraint algebra is formed by a Hamiltonian constraint and the generators of the kinematical symmetry. (iv) A Hamiltonian constraint is invariant under time reversal transformation. (v) A Hamiltonian constraint is an at most cubic polynomial function of canonical variables. (vi) There are no disconnected terms in a constraint algebra. The two forms are the same except for a slight difference in index contractions. The Hamiltonian constraint which was obtained in the previous paper and behaved oddly under time reversal symmetry can actually be transformed to one of them by a canonical change of variables. The two-fold uniqueness is shown up to the potential ambiguity of adding terms which vanish in the limit of pure gravitational physics.

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1 Introduction

Though quantum gravity has not yet fully been constructed, theoretical arguments based on the combination of the general relativity and quantum mechanics suggest that quantum gravitational fluctuations destroy precise spacetime measurements around the Planck energy [1]. This prompts the quest for a new quantum notion of spacetime in place of the classical one, which is based on continuous and smooth manifolds. From this perspective, the classical spacetime manifold is merely an infrared effective notion which emerges from underlying fundamental dynamics [2].

An interesting candidate of such a quantum notion is the fuzzy space, which describes a space with an algebra of functions on it* [3]-[8]. As discussed in [9]-[12], the dynamics of fuzzy spaces can be formulated as the rank-three tensor models, which have a rank-three tensor as their only dynamical variable. Then the rank-three tensor models can be regarded as a kind of quantum gravity.

In fact, tensor models have originally been proposed as a formulation of simplicial quantum gravity in dimensions higher than two [13, 14, 15]. The tensor models have later been generalized to describe topological lattice theories [16, 17] and the loop quantum gravity [18, 19, 20] by considering Lie-group valued indices. Interesting recent developments are the advent of the colored tensor models [21] and the subsequent discussions [22]-[42], which have presented improved formulations of tensor models. There have also been semi-classical studies of the rank-three tensor models by the present author [43]-[48], based on the interpretation in the previous paragraph. These semi-classical works have numerically shown the emergence of the (Euclidean) general relativity in the perturbations around the backgrounds representing various dimensional fuzzy flat spaces, which are classical solutions to certain fine-tuned rank-three tensor models.

The developments so far in tensor models have basically been dealing with the Euclidean cases. While field theories in flat Minkowski spacetimes can be connected to Euclidean ones by analytical continuation as a standard procedure, it seems a subtle problem whether this is also true in quantum gravity. In fact, the results of Causal Dynamical Triangulation suggest otherwise [49]. Another serious problem in the Euclidean tensor models is that their actions can freely be chosen as long as they respect the kinematical symmetries, and the tensor models have no predictive powers for possible future observations, while there may be chances for universality to save the situation [24, 27, 50].

* Noncommutative spaces [3, 4] are the special classes of fuzzy spaces, which are described by noncommutative associative algebra of functions. Nonassociative spaces [5]-[8] are also of physical interest.

In the previous paper [51], to overcome those problems in the Euclidean tensor models, the present author discussed the canonical formalism of the rank-three tensor model with "local" time. As in the canonical formalism of the general relativity [52, 53, 54], the form of the Hamiltonian constraint is strongly constrained by the requirement of the algebraic closure of the first class constraints. The previous paper has certainly provided a consistent Hamiltonian constraint, but it was not clarified whether it was unique or not. Moreover, it seemed problematic that the Hamiltonian constraint had a form which behaved oddly under the time reversal transformation.

The purpose of the present paper is to write down all the allowed forms of a Hamiltonian constraint under the following physically reasonable assumptions: (i) A Hamiltonian constraint has one index. (ii) The kinematical symmetry is given by an orthogonal group. (iii) A consistent first class constraint algebra is formed by a Hamiltonian constraint and the generators of the kinematical symmetry. (iv) A Hamiltonian constraint is invariant under time reversal transformation. (v) A Hamiltonian constraint is an at most cubic polynomial function of canonical variables. (vi) There are no disconnected terms in a constraint algebra.

The discussions will be based on brute-force analysis. The general form of a Hamiltonian constraint respecting the assumptions except for the algebraic ones in the above will be written down, and then the algebraic assumptions will be imposed by explicitly computing the Poisson brackets for the general form. It will be shown that only two forms of a Hamiltonian constraint, which have a slight difference in index contractions, satisfy the assumptions. It will also be found that, after a canonical change of variables, the Hamiltonian constraint which was obtained in the previous paper is indeed identical to one of them. The two-fold uniqueness above will be shown up to the potential ambiguity of adding terms which will vanish in the limit of pure gravitational physics.

This paper is organized as follows. In Section 2, graphical representation will be introduced to efficiently carry out the computations, which would otherwise become very much cumbersome. It will be explained that the graphical representation introduced in this section is not completely precise, but will give some necessary conditions, which will turn out to drastically reduce the possibilities of a Hamiltonian constraint. In Section 3, some general properties of a Hamiltonian constraint will be discussed. Then all the possible terms of a Hamilton constraint respecting the above assumptions except for the algebraic ones will be written down by using the graphical representation. In Section 4, the Poisson bracket of the Hamiltonian constraints with the general terms will be computed by using the graphs. In Section 5, the algebraic assumptions will be imposed, and the solutions satisfying the necessary conditions will be obtained. In Section 6, to find the full solutions, signatures will be introduced to the graphical representation to precisely represent the Poisson bracket, and the complete conditions will

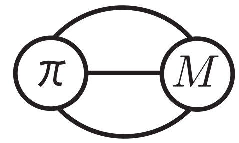

Figure 1: The graphical representation of $c_1\pi_{abc}M_{abc} + c_2\pi_{abc}M_{bac}$ with $c_1 + c_2 = 1$.

be obtained. It will be shown that there exist only two forms for a Hamiltonian constraint. Section 7 will be devoted to the summary and future prospects.

2 Graphical representation

The canonical variables of the rank-three tensor models are assumed to be given by M_{abc} $(a, b, c = 1, 2, \ldots, N)$ and its conjugate momentum π_{abc}. They are assumed to satisfy the generalized Hermiticity condition,

$$\begin{aligned} &M_{abc} = M_{bca} = M_{cab} = M^*_{bac} = M^*_{acb} = M^*_{cba}, \\ &\pi_{abc} = \pi_{bca} = \pi_{cab} = \pi^*_{bac} = \pi^*_{acb} = \pi^*_{cba}, \end{aligned} \tag{1}$$

where $*$ denotes the complex conjugation. The Poisson brackets of the canonical variables are assumed to be given by

$$\begin{aligned} &\{M_{abc}, \pi_{def}\} = \delta_{abc,def} \equiv \delta_{ad}\delta_{be}\delta_{cf} + \delta_{ae}\delta_{bf}\delta_{cd} + \delta_{af}\delta_{bd}\delta_{ce}, \\ &\{M_{abc}, M_{def}\} = \{\pi_{abc}, \pi_{def}\} = 0, \end{aligned} \tag{2}$$

which respect the generalized Hermiticity (1).

The fact that the canonical variables have three indices tends to make computations cumbersome and inefficient. To avoid this, let me introduce graphical representation as follows. As shown in an example in Figure 1, a blob represents M_{abc} or π_{abc}, and connected lines represent contractions of the indices. While the Hermiticity condition (1) assures the invariance under the cyclic rotations of the indices, which correspond to the rotations of a blob, the relative orders of the indices, such as the distinction between $\pi_{abc}M_{abc}$ and $\pi_{abc}M_{bac}$, are relevant. In the graphical representation, this kind of interchange of the order of the indices would generate crossing of the lines. In Sections 3, 4 and 5, to avoid such entanglement of the lines, it will be assumed that the graphical representation does not care the order: the graph in Figure 1, for example, represents either $\pi_{abc}M_{abc}$ or $\pi_{abc}M_{bac}$, or even a linear combination of them, the coefficients of which are assumed to add up to 1.

In general, this ambiguous treatment of the order of the indices will generate ambiguity in the computations of the Poisson brackets. As an illustration, let me consider the following

simple computation of a Poisson bracket,

$$\{c_1\pi_{abc}M_{abc} + c_2\pi_{abc}M_{bac}, \pi_{abc}\pi_{abc}\} = 6c_1\pi_{abc}\pi_{abc} + 6c_2\pi_{abc}\pi_{bac}. \tag{3}$$

Let me assume that a graph does not care the order of the indices so that each value of c_i is not determined, but the sum is fixed by $c_1 + c_2 = 1$. Though the result in the right-hand side of (3) is certainly ambiguous, this ambiguity can be deleted by assuming π_{abc} (and also M_{abc} in general) to be real. This is because, from the reality assumption and the Hermiticity (1), π_{abc} (and M_{abc}) become totally symmetric with respect to the indices, and the right-hand side of (3) adds up to $6(c_1+c_2)\pi_{abc}\pi_{abc}$, which is not ambiguous. This trick of imposing the reality assumption on the results of Poisson brackets to obtain unambiguous results is not always applicable, but can be applied to simple graphs, and will drastically simplify the discussions of this paper.

Thus, in Sections 3, 4 and 5, I will assume the reality of M_{abc} and π_{abc} for the results of Poisson brackets. In fact, one can easily check, for each of the following computations, that the results are not ambiguous, if the reality assumption is imposed, even though a graph does not care the order of the indices. However, if this reality assumption is taken, only some necessary conditions for the algebraic consistency will be obtained, since the algebraic consistency is considered only on the slice of the real values of the canonical variables. Therefore, after the possibilities of a Hamiltonian constraint have drastically been reduced by the necessary conditions, the complete treatment which cares the order of the indices will follow in Section 6.

3 The general form of a Hamiltonian constraint

The kinematical symmetry of the rank-three tensor model is assumed to be given by the orthogonal group $O(N)$, which is represented by the vector representation on the indices of M_{abc} and π_{abc}. With the canonical variables, the generators are expressed as

$$\mathcal{D}_{ab} = \frac{1}{2}\left(\pi_{acd}M_{bcd} - \pi_{bcd}M_{acd}\right). \tag{4}$$

They satisfy

$$\{D(V^1), D(V^2)\} = D([V^1, V^2]), \tag{5}$$

where

$$D(V) \equiv V_{ab}\mathcal{D}_{ab} \tag{6}$$

with an anti-symmetric real matrix V_{ab}, and [,] denotes a matrix commutator.

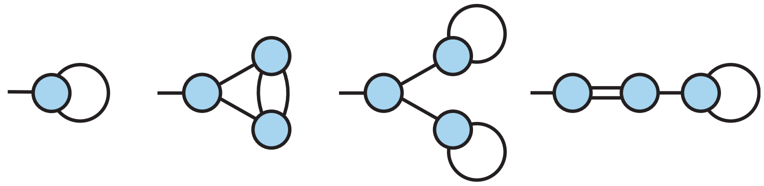

Figure 2: The topological structures of the possible terms in $\mathcal{H}_a$. The blobs represent π_{abc} or M_{abc}.

As discussed in the previous paper [51], a Hamiltonian constraint should have an index as $\mathcal{H}_a$. It is also assumed that $\mathcal{H}_a$ is a polynomial function of the canonical variables, whose indices should be contracted in pairs except for the one corresponding to the index a of $\mathcal{H}_a$. Then, since the inner indices are contracted to be invariant under the orthogonal group symmetry, the group transformation applies only to the index a of $\mathcal{H}_a$, and therefore

$$\{D(V), H(T)\} = H(VT) \tag{7}$$

is satisfied, where

$$H(T) \equiv T_a \mathcal{H}_a \tag{8}$$

with a real vector T_a.

Another assumption is that the terms which compose $\mathcal{H}_a$ be represented by connected graphs. This assumption is physically required, because, otherwise, the dynamics of the tensor models would become non-local on an emergent space. By also assuming $\mathcal{H}_a$ be at most cubic in canonical variables, the possible topological structures of the graphs representing the terms in $\mathcal{H}_a$ can be summarized as in Figure 2.

The assumption of the time reversal symmetry, that $\mathcal{H}_a$ be invariant under $\pi_{abc} \to -\pi_{abc}$, requires that there are even numbers of π_{abc} in each term. Then, by ignoring the index ordering as explained in the previous section, all the possible terms in $\mathcal{H}_a$ can be listed as in Figure 3.

4 Poisson bracket of Hamiltonian constraints

Based on the discussions in the previous section, the general form of a Hamiltonian constraint can be expressed as

$$\mathcal{H}_a = \sum_{i=1}^{11} d_i \, G_a^i, \tag{9}$$

where d_i are real numerical coefficients, and G_a^i are the terms represented by the graphs in Figure 3. In this section, the Poisson bracket of the Hamiltonian constraints in the general form (9) will be computed.

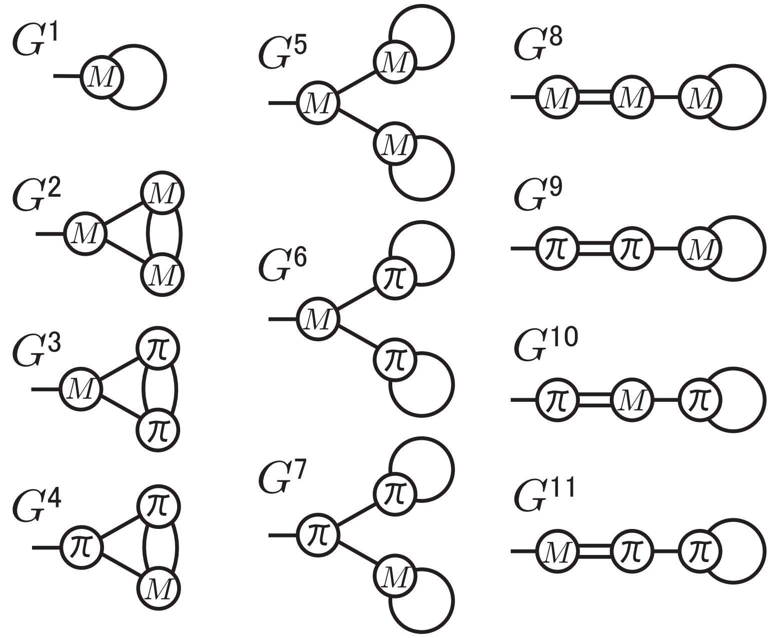

Figure 3: All the possible terms in $\mathcal{H}_a$ which respect the time reversal symmetry.

Let me introduce the notion of π-degree, which counts the number of π_{abc} in each term. For example, the graphs G^1 and G^3 have the π-degree 0 and 2, respectively. This π-degree is additive in multiplication, and a Poisson bracket can be considered to have the π-degree -1, since the number of π_{abc} is reduced by one by computing a non-vanishing Poisson bracket. Since the π-degree is a conserved quantity with this assignment, and $\mathcal{H}_a$ is composed of the terms with the π-degree 0 or 2, the Poisson bracket of the Hamiltonian constraints is given by

$$\{H(T^1), H(T^2)\} = (\text{terms with the } \pi\text{-degree } 3) + (\text{terms with the } \pi\text{-degree } 1), \tag{10}$$

where the former terms in the right-hand side come from the Poisson brackets of the terms with the π-degree 2, while the latter terms from those of the terms with the π-degree 2 and 0. Therefore, one can separately discuss the closure condition of the constraint algebra by classifying the terms according to the π-degree. So let me first consider the terms with the π-degree 2 in $\mathcal{H}_a$ by putting $d_1 = d_2 = d_5 = d_8 = 0$ to suppress the terms with the π-degree 0. The allowed values of d_1, d_2, d_5, d_8 will be discussed later in Sections 5 and 6.

The computations of the Poisson brackets can efficiently be carried out by using the graphical representation. As an example, the computation of $\{G^3(T^1), G^3(T^2)\}$, where $G^i(T) \equiv T_a G^i_a$, is illustrated in Figure 4. From the first to the second line of the figure, pairs of M_{abc} and π_{abc} are deleted from the graphs, either one from each graph, and, in the third line, the pair of the open graphs in the second line are connected by gluing the open lines in all the possible ways without caring the index ordering. From the third to the last line, the graph symmetric

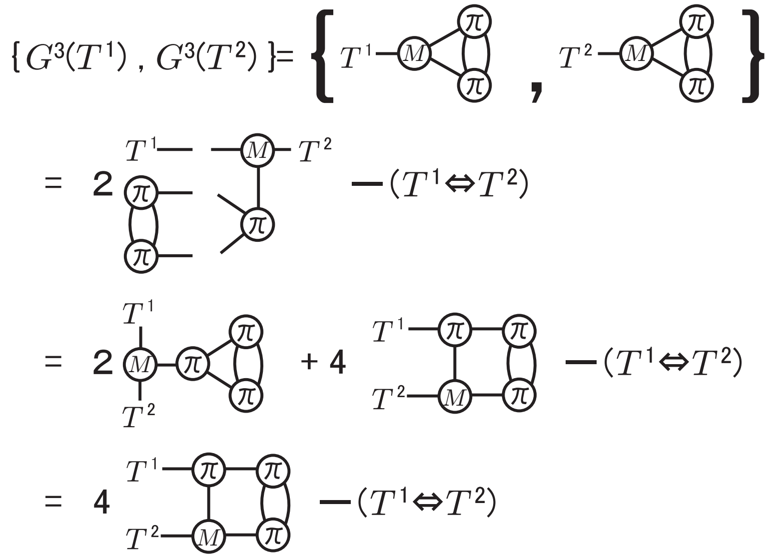

Figure 4: The computation of $\{G^3(T^1), G^3(T^2)\}$ by the graphs.

under the interchange $T^1 \leftrightarrow T^2$ has been canceled by the correspondence in $-(T^1 \leftrightarrow T^2)$ in the figure[†]. In the computation, it is important to take correctly into account the multiplicity and also the signature coming from which of $\{M_{abc}, \pi_{def}\}$ or $\{\pi_{abc}, M_{def}\}$ is computed.

The Poisson brackets of the other graphs can be computed in the same manner. The result

[†] Note that, as explained in Section 2, M_{abc} and π_{abc} are totally symmetric with respect to the indices under the reality assumption.

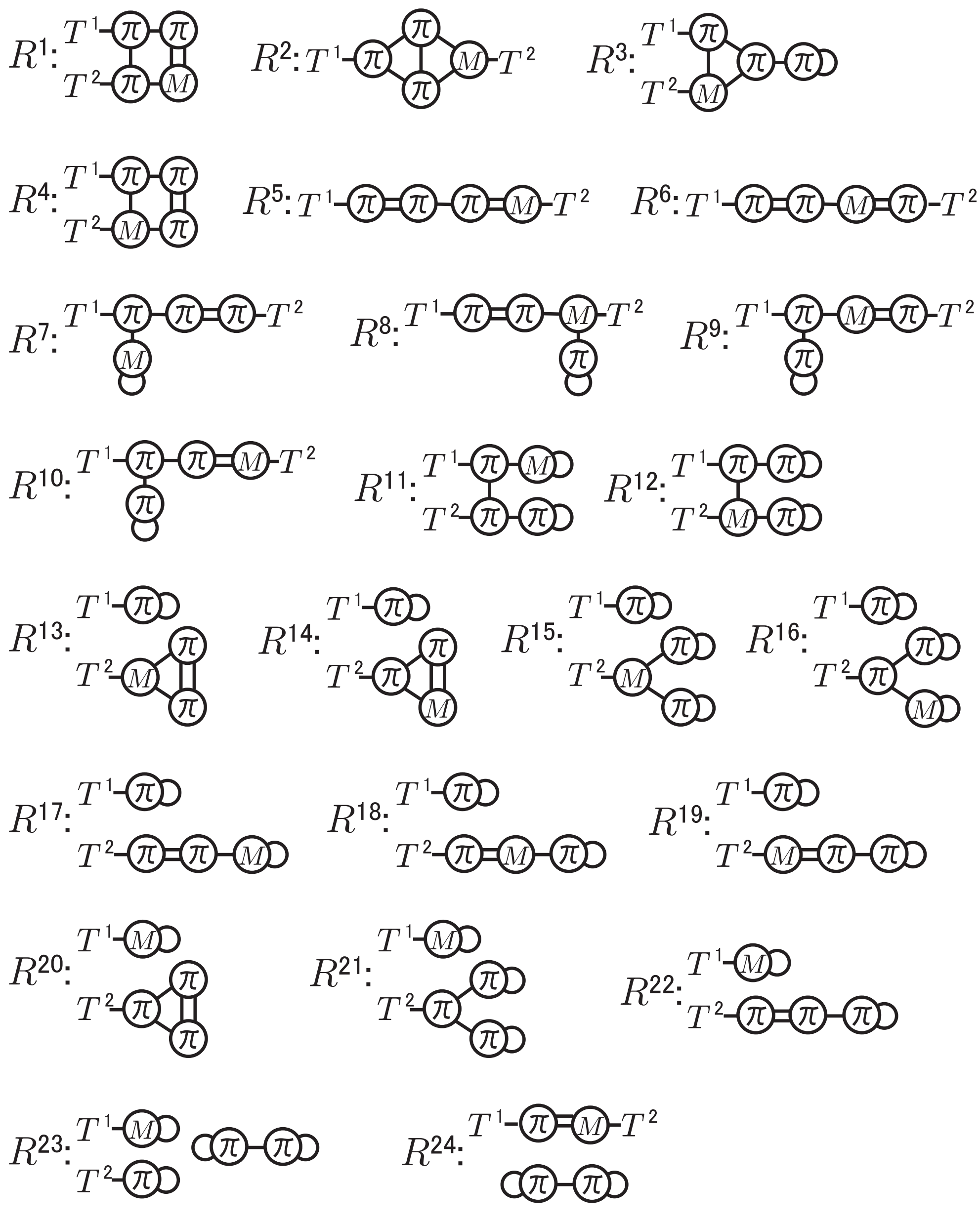

Figure 5: All the graphs which are generated from $\{H(T^1), H(T^2)\}$.

is

$$
\begin{aligned}
&\{H(T^1), H(T^2)\} = (d_4)^2 R^1 + 4d_3 d_4 R^2 + (4d_3 d_{11} + 4d_3 d_{10}) R^3 + 4(d_3)^2 R^4 \\
&\quad + (-d_3 d_4 + 2d_4 d_{11} + d_4 d_9 + 2d_3 d_{11} + 4d_3 d_9 + (N+2) d_9 d_{11}) R^5 \\
&\quad + (-d_3 d_4 + 2d_4 d_{10} + 3d_4 d_9 + 2d_3 d_{10} + (N+2) d_9 d_{10}) R^6 \\
&\quad + (2d_3 d_9 - 4(d_9)^2 - 2d_4 d_7 - 2d_3 d_7 - (N+2) d_7 d_9) R^7 \\
&\quad + (2d_3 d_9 + 2d_9 d_{11} + 2d_9 d_{10} + 4d_4 d_6 + 4d_3 d_6 + 2(N+2) d_6 d_9) R^8 \\
&\quad + (2d_{10} d_{11} + 2(d_{10})^2 + 3d_4 d_7 + (N+2) d_7 d_{10}) R^9 \\
&\quad + (2(d_{11})^2 + 2d_{10} d_{11} + d_4 d_7 + 4d_3 d_7 + (N+2) d_7 d_{11}) R^{10} \\
&\quad + (2d_9 d_{11} + 2d_9 d_{10} - (N+2)(d_7)^2 - 2d_4 d_7 - 2d_7 d_{11} - 2d_7 d_{10} - 4d_7 d_9) R^{11} \\
&\quad + (2(N+2) d_6 d_7 + 2d_3 d_7 + 2d_7 d_{11} + 2d_7 d_{10} + 4d_3 d_6 + 4d_6 d_{11} + 4d_6 d_{10}) R^{12} \\
&\quad + d_3 d_{11} R^{13} - d_4 d_{10} R^{14} + (4(d_6)^2 + d_6 d_{11}) R^{15} \\
&\quad + (-(d_7)^2 + 2d_6 d_7 + d_7 d_{11} - d_7 d_9 - 2d_6 d_9) R^{16} + (-d_9 d_{10} - (d_9)^2 + d_3 d_7 - d_7 d_9) R^{17} \\
&\quad + (-d_4 d_6 + 2d_6 d_{10}) R^{18} + ((d_{11})^2 + d_{10} d_{11} - d_4 d_6 + 2d_6 d_{11}) R^{19} + d_3 d_9 R^{20} \\
&\quad + (-(d_7)^2 + d_6 d_9) R^{21} + (d_3 d_7 - d_7 d_9) R^{22} + d_6 d_7 R^{23} + ((d_{11})^2 + d_6 d_{11} - d_6 d_{10}) R^{24} \\
&\quad - (T^1 \leftrightarrow T^2),
\end{aligned} \tag{11}
$$

where R^i's are defined in Figure 5.

5 Solutions to the consistency of the constraint algebra

An assumption of this paper is that $\mathcal{H}_a$ and $\mathcal{D}_{ab}$ form a consistent first class constraint algebra. From (5) and (7), the Poisson brackets containing $\mathcal{D}_{ab}$ generally satisfy this assumption. On the other hand, the Poisson bracket of $H(T)$'s obtained in (11) has many unwanted terms and the requirement of the algebraic consistency will strongly restrict the allowed values of d_i.

First of all, the assumption of the graphical connectivity of the terms existing in the algebra requires that the terms represented by R^i ($i = 13, 14, \ldots, 24$) should not appear in (11), since these graphs are disconnected as listed in Figure 5.

Next, as one can see in Figure 5, $R^1 - (T^1 \leftrightarrow T^2)$ and $R^{24} - (T^1 \leftrightarrow T^2)$ contain $\mathcal{D}_{ab}$ as their parts. This is also true for $R^5 - R^6$ and $R^9 - R^{10}$. Therefore these terms in (11) do not violate the consistency of the constraint algebra, and are allowed.

The other graphs or combinations of the graphs in Figure 5 do not contain the same structures as the graphs of $\mathcal{H}_a$ in Figure 3 or of $\mathcal{D}_{ab}$. Therefore all the coefficients of these terms in (11) must vanish. The conditions for this vanishing to hold for general N are given

by

$$
\begin{aligned}
d_3d_4 &= 0,\\
d_3(d_{10}+d_{11}) &= 0,\\
(d_3)^2 &= 0,\\
-d_3d_4 + d_4(d_{10}+d_{11}) + 2d_4d_9 + d_3(d_{10}+d_{11}) + 2d_3d_9 &= 0,\\
d_9(d_{10}+d_{11}) &= 0,\\
d_3d_9 - 2(d_9)^2 - d_4d_7 - d_3d_7 &= 0,\\
d_7d_9 &= 0,\\
d_3d_9 + d_9(d_{10}+d_{11}) + 2d_4d_6 + 2d_3d_6 &= 0,\\
d_6d_9 &= 0,\\
(d_{10}+d_{11})^2 + 2(d_3+d_4)d_7 &= 0,\\
d_7(d_{10}+d_{11}) &= 0,\\
(d_9-d_7)(d_{10}+d_{11}) - d_4d_7 - 2d_7d_9 &= 0,\\
(d_7)^2 &= 0,\\
d_3d_7 + d_7(d_{10}+d_{11}) + 2d_3d_6 + 2d_6(d_{10}+d_{11}) &= 0,\\
d_6d_7 &= 0.
\end{aligned}
\tag{12}
$$

Here I have taken into account the allowance explained in the previous paragraph. These equations determine

$$
d_3 = d_7 = d_9 = d_{10} + d_{11} = 0, \tag{13}
$$

while there exist two non-vanishing cases for d_4 and d_6 as[‡]

$$
\text{(i) } d_4 \neq 0,\ \ d_6 = 0, \tag{14}
$$

$$
\text{(ii) } d_4 = 0,\ \ d_6 \neq 0. \tag{15}
$$

The case (ii) in (15) is not appropriate, since one can easily show that this case contradicts the absence of the disconnected terms explained above.

On the other hand, for the case (i), by substituting (14) into (11), one finds that, for the absence of the disconnected terms, $d_{10} = d_{11} = 0$ is also required. Then the Poisson bracket (11) becomes $R^1 - (T^1 \leftrightarrow T^2)$. Since $R^1 - (T^1 \leftrightarrow T^2)$ contains $\mathcal{D}_{ab}$ as its part, the case (i) is the primary candidate for a consistent constraint algebra. So, the only physically consistent solution is the case (i), and the terms with the π-degree 2 in $\mathcal{H}_a$ are exhausted by G^4.

‡ $d_4 = d_6 = 0$ is not appropriate, since then the Hamiltonian constraint contains only the terms proportional to $\mathcal{D}_{ab}$.

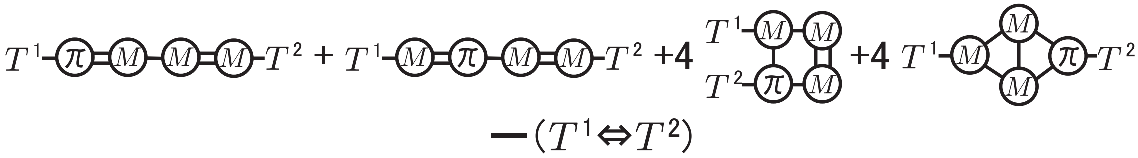

Figure 6: The result of $\{G^4(T^1), G^2(T^2)\} + \{G^2(T^1), G^4(T^2)\}$.

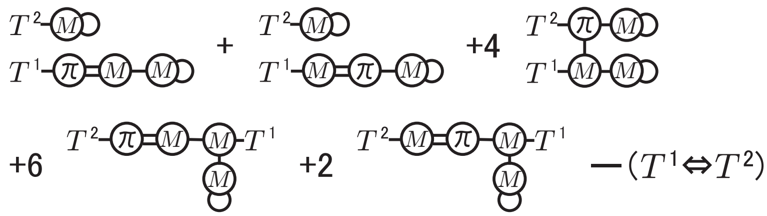

Figure 7: The result of $\{G^4(T^1), G^5(T^2)\} + \{G^5(T^1), G^4(T^2)\}$.

Let me next study the terms with the π-degree 1 in the right-hand side of (10). These terms come from the Poisson brackets of G^4 and the graphs, G^1, G^2, G^5, G^8, with the π-degree 0. Since G^1 and the graphs, G^2, G^5, G^8, have different degrees of the canonical variables, the two kinds of graphs can be discussed separately. Let me consider only G^2, G^5, G^8 in the following discussions, leaving G^1 for later discussions in Section 6.

The Poisson brackets, $\{G^4(T^1), G^i(T^2)\} + \{G^i(T^1), G^4(T^2)\}$ $(i = 2, 5, 8)$, can be computed graphically as before, and the results are given in Figures 6, 7 and 8. Since each of these figures contains at least one graph, which is not contained in the other figures and contains no parts identical to $\mathcal{D}_{ab}$, the only solution to the algebraic consistency is the vanishing one, $d_2 = d_5 = d_8 = 0$.

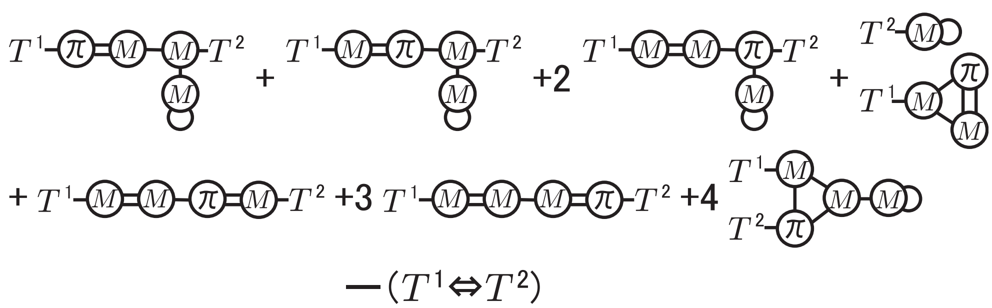

Figure 8: The result of $\{G^4(T^1), G^8(T^2)\} + \{G^8(T^1), G^4(T^2)\}$.

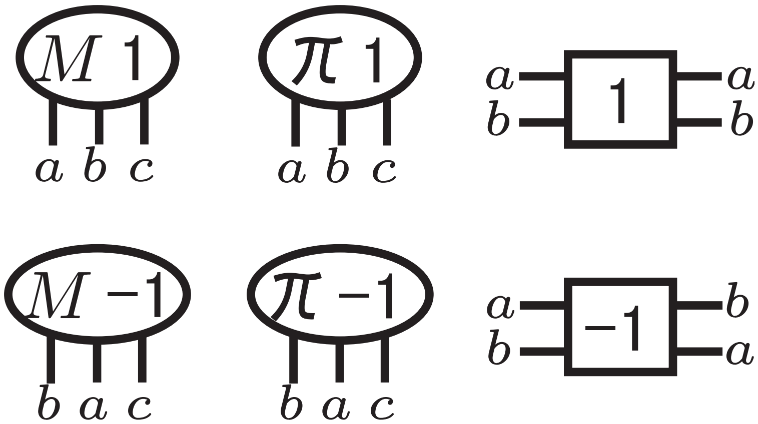

Figure 9: In the leftmost two figures, M_{abc} is represented in two ways, in which the order of the indices depends on the signature. In the central two figures, π_{abc} is also represented in the two ways. In the rightmost upper figure, the left couple of the lines are connected to the right ones in parallel, while, in the rightmost lower figure, they are connected after being twisted.

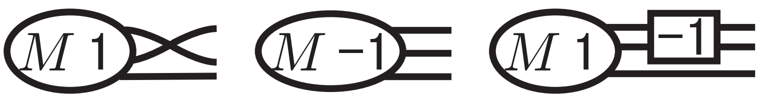

Figure 10: The new graphs can be used to unwind the entangled lines. The three graphs in the figure have the same connectivity of the lines.

6 Computations respecting the order of the indices

The previous section concludes that the Hamiltonian constraint can only have the terms represented by G^4 and possibly by G^1. As explained in Section 2, the graphical representation in Sections 3, 4 and 5 is not supposed to unambiguously specify the index contractions of the canonical variables, and cannot fully determine a Hamiltonian constraint. To obtain the full solutions, one has to keep precise track of the connections of the lines in a graph, which will generally generate a complicated entangled graph. To avoid this entanglement, let me introduce signatures ± 1 to the graphs as in Figure 9. These new graphs unwind the entanglement of the lines, as the examples in Figure 10.

By considering all the possible ways of the index contractions of the term represented by G^4, one can find that four of them are independent and can be represented by a graph with two signatures, E^{ij} $(i, j = \pm 1)$, defined in Figure 11. The Hamiltonian constraint, which was ambiguously represented by G^4 in the previous sections, can generally have a form,

$$\mathcal{H}_a = c_{i,j} E_a^{ij}, \tag{16}$$

where $c_{i,j}$ $(i, j = \pm 1)$ are real numerical coefficients. The computations of the Poisson brackets

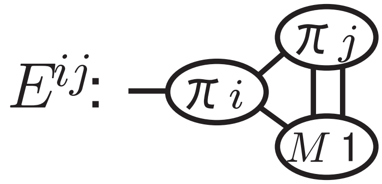

Figure 11: The graphical representation of E^{ij} $(i, j = 1, 2)$.

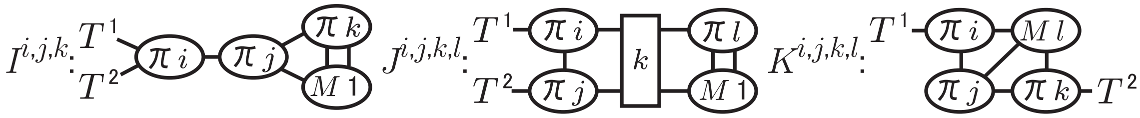

Figure 12: The graphical representation of I, J, K.

of E^{ij}_a can be carried out in the same manner as in the previous sections, but this time the connections of the lines are precisely taken into account. Then one obtains

$$\{T^1_a E^{ij}_a, T^2_b E^{kl}_b\} = I^{i,-jk,l} + J^{-i,-j,k,l} + J^{-i,j,-k,l} + J^{i,k,1,-jl} + K^{-i,-j,k,l} + K^{-i,j,k,-l} - (T^1 ij \leftrightarrow T^2 kl), \tag{17}$$

where I, J, K are defined in Figure 12. It is easy to show that these I, J, K are transformed by the interchange $T^1 \leftrightarrow T^2$ to

$$\begin{aligned} I^{i,j,k} &\to I^{-i,j,k}, \\ J^{i,j,k,l} &\to J^{-j,-i,-k,l}, \\ K^{i,j,k,l} &\to K^{-k,-j,-i,-l}. \end{aligned} \tag{18}$$

Then, from (16), (17) and (18), one obtains

$$\begin{aligned} \{H(T^1), H(T^2)\} &= (c_{i,l}c_{-lj,k} - c_{-i,l}c_{-lj,k}) I^{i,j,k} + (c_{-i,-j}c_{k,l} + c_{-i,j}c_{-k,l} - c_{j,i}c_{-k,l} - c_{j,-i}c_{k,l}) J^{i,j,k,l} \\ &+ c_{i,l}c_{j,-kl} J^{i,j,1,k} - c_{-j,l}c_{-i,-kl} J^{i,j,-1,k} + (c_{-i,-j}c_{k,l} + c_{-i,j}c_{k,-l} - c_{k,j}c_{-i,-l} - c_{k,-j}c_{-i,l}) K^{i,j,k,l}. \end{aligned} \tag{19}$$

By comparing Figures 11 and 12, one finds that $c_{j,k} I^{i,j,k}$ contains the Hamiltonian constraint (16) as its part. And, from the expression (4), one sees also that $I^{i,1,-1} - I^{i,-1,-1}$ and $J^{i,j,1,-1} - J^{i,j,-1,-1}$ contain $\mathcal{D}_{ab}$. Therefore, from (19), the condition for the algebraic closure of the constraint algebra is obtained as

$$\begin{aligned} & c_{i,l}c_{-lj,k} - c_{-i,l}c_{-lj,k} = \lambda_i c_{j,k} + \tilde{\lambda}_i (\delta_{j,1}\delta_{k,-1} - \delta_{j,-1}\delta_{k,-1}), \\ & c_{-i,-j}c_{k,l} + c_{-i,j}c_{-k,l} - c_{j,i}c_{-k,l} - c_{j,-i}c_{k,l} + c_{i,m}c_{j,-lm}\delta_{k,1} - c_{-j,m}c_{-i,-lm}\delta_{k,-1} \\ & \qquad\qquad\qquad\qquad\qquad\qquad = \lambda_{ij}\delta_{k,1}\delta_{l,-1} - \lambda_{ij}\delta_{k,-1}\delta_{l,-1}, \\ & c_{-i,-j}c_{k,l} + c_{-i,j}c_{k,-l} - c_{k,j}c_{-i,-l} - c_{k,-j}c_{-i,l} = 0, \end{aligned} \tag{20}$$

where λ_i, $\tilde{\lambda}_i$, and λ_{ij} are the real numbers which express the proportionality to the Hamiltonian constraint or $\mathcal{D}_{ab}$.

Because of the antisymmetry of the left-hand side of the first equation of (20) under $i \to -i$,

$$(\lambda_i + \lambda_{-i})c_{j,k} + (\tilde{\lambda}_i + \tilde{\lambda}_{-i})(\delta_{j,1}\delta_{k,-1} - \delta_{j,-1}\delta_{k,-1}) = 0 \tag{21}$$

must be satisfied. Since a Hamiltonian constraint and $\mathcal{D}_{ab}$ should be independent as a physical requirement, the former and the latter terms in (21) must vanish independently. Therefore

$$\begin{aligned} \lambda_{-i} &= -\lambda_i, \\ \tilde{\lambda}_{-i} &= -\tilde{\lambda}_i, \end{aligned} \tag{22}$$

can be assumed.

From the algebraic closure of $\mathcal{D}_{ab}$ expressed as (5) and (7), it is obvious that one can freely add $\pi_{abc}\mathcal{D}_{bc}$, which can be expressed in the form of the right-hand side of (16), to a Hamiltonian constraint without violating the closure of a constraint algebra. Therefore one can reduce the number of free parameters of the solutions to (20) by fixing this free addition. From (4), (16) and Figure 11, this addition corresponds to a shift of $c_{i,j}$ by

$$\delta c_{i,j} = \epsilon(\delta_{i,1}\delta_{j,-1} - \delta_{i,-1}\delta_{j,-1}), \tag{23}$$

where ϵ is an infinitesimal parameter. By substituting (23) into the first equation of (20), one finds that (23) is equivalent to[§]

$$\delta\lambda_i = 2\epsilon(\delta_{i,1} - \delta_{i,-1}). \tag{24}$$

Therefore, from (22) and (24), one can safely assume

$$\lambda_i = 0 \tag{25}$$

without loss of generality, with the allowance that one may freely add (23) to the solutions.

Then the first equation of (20) gives

$$\begin{aligned} &[(c_{1,1} - c_{-1,1}) + (c_{1,-1} - c_{-1,-1})](c_{1,k} + c_{-1,k}) = 0, \\ &(c_{1,1} - c_{-1,1})c_{-j,1} + (c_{1,-1} - c_{-1,-1})c_{j,1} = 0. \end{aligned} \tag{26}$$

Here the first equation of (26) has been derived by summing over $j = \pm 1$ to delete $\tilde{\lambda}_i$, and the second of (26) by substituting $k = 1$. The second equation of (26) can be used to delete $c_{1,-1} - c_{-1,-1}$ from the first equation of (26) to obtain

$$(c_{1,1} - c_{-1,1})^2(c_{1,k} + c_{-1,k}) = 0. \tag{27}$$

[§] $\tilde{\lambda}_i$ is also shifted.

This shows that $c_{1,1} = c_{-1,1}$ or $c_{1,k} = -c_{-1,k}$ $(k = \pm 1)$. Substituting the first case into (26) leads to $c_{1,-1} = c_{-1,-1}$ or $c_{i1} = 0$ and $(c_{1,-1} - c_{-1,-1})(c_{1,-1} + c_{-1,-1}) = 0$. Thus the solutions to the first equation of (20) can be classified into the following two cases,

$$
\begin{aligned}
&\text{(i) } c_{1,k} = c_{-1,k} \ (k = \pm 1), \\
&\text{(ii) } c_{1,k} = -c_{-1,k} \ (k = \pm 1).
\end{aligned}
\tag{28}
$$

By substituting the case (i) of (28) into the second equation of (20), one obtains

$$
c_{1,1} c_{1,-1} = 0. \tag{29}
$$

Therefore there are two non-vanishing solutions,

$$
\begin{aligned}
&\text{(i-1) } c_{1,1} = c_{-1,1} \neq 0, \ c_{1,-1} = c_{-1,-1} = 0, \\
&\text{(i-2) } c_{1,1} = c_{-1,1} = 0, \ c_{1,-1} = c_{-1,-1} \neq 0.
\end{aligned}
\tag{30}
$$

On the other hand, by substituting the case (ii) of (28) into the second equation of (20), one obtains

$$
(c_{1,1})^2 = 0. \tag{31}
$$

The solution, $c_{1,1} = c_{-1,1} = 0, c_{1,-1} = -c_{-1,-1}$, just represents $\pi_{abc}\mathcal{D}_{bc}$, and is physically rejected as a Hamiltonian constraint. The third equation of (20) is always satisfied. It can be checked that the two solutions in (30) actually satisfy all the equations in (20).

Thus the conclusion is that, from (30), the forms of a Hamiltonian constraint which satisfy the algebraic consistency are exhausted by

$$
\begin{aligned}
\mathcal{H}_a &= \pi_{a(bc)} \pi_{bde} M_{cde}, \\
\mathcal{H}_a &= \pi_{a(bc)} \pi_{bde} M_{ced},
\end{aligned}
\tag{32}
$$

where ( ) denotes symmetrization of the indices, $\pi_{a(bc)} \equiv (\pi_{abc} + \pi_{acb})/2$.

In the both cases of (32), one obtains,

$$
\{H(T^1), H(T^2)\} = D([\tilde{T}^1, \tilde{T}^2]), \tag{33}
$$

where $\tilde{T}^i$ are the symmetric matrices defined by

$$
\tilde{T}^i{}_{bc} = T^i_a \pi_{a(bc)}. \tag{34}
$$

As was discussed in the previous paper [51], the constraint algebra (5), (7) and (33) would reproduce the first class constraint algebra of the canonical formalism of the general relativity

for the Minkowski signature in the pointwise limit¶. In the derivation, the dependence of $\tilde{T}^i$ on the canonical variables as in (34) played the essential roles.

Let me next discuss the possibility of adding a term represented by G^1 to $\mathcal{H}_a$. The Poisson bracket of Hamiltonian constraints will be changed by

$$\{H(T^1), T^2_a M_{abb}\} + \{T^1_a M_{abb}, H(T^2)\} = -2(T^1_a T^2_b - T^1_b T^2_a)\pi_{a(cd)} M_{b(cd)}. \tag{35}$$

The right-hand side looks very similar to but is not the same as $\mathcal{D}_{ab}$ because of the symmetrization of the contracted indices. Therefore the algebra will not close, and one cannot add the term represented by G^1 to $\mathcal{H}_a$.

In the previous paper, another form of a Hamiltonian constraint has been presented. Indeed one can easily check that the previous form, except for the linear term, can be obtained by applying a canonical transformation $M \to (M \pm \pi)\sqrt{2}$, $\pi \to (\pi \mp M)/\sqrt{2}$ to the first one in (32). The absence of the linear term is because it corresponds to the term π_{abb} in the parameterization of this paper, and has been rejected due to its violation of the time reversal symmetry.

7 Summary and future prospects

In the previous paper [51], Hamiltonian formalism of the rank-three tensor model has been proposed, in which "local" time is consistently incorporated by a first class constraint algebra. A consistent Hamiltonian constraint was presented in the paper, but it was not clear whether there were other possibilities or not. Moreover, it behaved oddly under the time reversal transformation.

To solve these issues, this paper has given the thorough discussions on the allowed forms of a Hamiltonian constraint, assuming the physically reasonable conditions (i)-(vi) listed in Section 1, among which (iv) imposes the time reversal symmetry. The closure condition of a constraint algebra strongly restricts the allowed forms, and it has been shown that there exist only two consistent forms, which have a slight difference in index contractions. The Hamiltonian constraint obtained in the previous paper can indeed be transformed to one of them by a canonical change of variables.

One must be cautious about how far the two-fold uniqueness has been shown in this paper. In fact, in passing from the unordered computations to the ordered ones, the possibilities for

¶It seems impossible to reproduce the first class constraint algebra of general relativity in the Euclidean signature, though there are no proofs for no-go.

certain linear combinations of terms to exist in a Hamiltonian constraint have been ignored. To explain more concretely, let me consider the ordered graphs $\tilde{G}^{\alpha\, i}$, where α is a new label, corresponding to the unordered graphs G^i in Figure 3. Then, since the unordered computations do not recognize the difference between $\tilde{G}^{\alpha\, i}$ and $\tilde{G}^{\beta\, i}$, the unordered computations of Poisson brackets remain the same even if a linear combination $\sum_{\alpha,i} \tilde{c}_{\alpha\, i} \tilde{G}^{\alpha\, i}$ with $\sum_\alpha \tilde{c}_{\alpha\, i} = 0$ is added to a Hamiltonian constraint. This means that it is not right in Section 6 to assume that there only exist the ordered graphs corresponding to G^4 in a Hamiltonian constraint, based on the unordered computations. In this sense, a complete proof of the two-fold uniqueness has not been given in this paper.

On the other hand, from the prospective of gravitational physics, the uniqueness has been proven. As discussed in [43]-[48], the gravitational degrees of freedom are described by the commutative part of fuzzy spaces. This part corresponds to the totally symmetric part of the tensors. Therefore, for the purpose of quantum gravity, one could formulate the rank-three tensor models with totally symmetric real tensors instead of Hermitian complex-valued tensors of this paper. Then the unordered computations are enough and a Hamiltonian constraint can only have G^4 and G^1. For totally symmetric tensors, the two final forms in (32) are identical, and the uniqueness is literally true. Another new thing is that the right-hand side of (35) is proportional to $\mathcal{D}_{ab}$, and therefore G^1 is also allowed to exist in a Hamiltonian constraint.

Once a dynamical system is defined by a first class constraint algebra, the next future questions would be how to quantize it and what is its dynamics. In the present motivation for the rank-three tensor model, highly interesting would be the question of whether there exist the classical regimes in which the system behaves as if there is a spacetime respecting locality - *emergent spacetime* [2]. The two-fold uniqueness of the Hamiltonian constraint shown in this paper will simplify the future study of this question in the rank-three tensor model.

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